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On E-Pólya frequency functions

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ABSTRACT

Let E be a subset of $\mathbb{R}$. We say that $f:\mathbb{R}\to\mathbb{R}$ is an E-Pólya frequency function if the kernel $f(u-x)$ is totally positive on $E\times\mathbb{R}$ and f is integrable on $\mathbb{R}$. In this paper, we show that under some conditions on E, an E-Pólya frequency function is a Pólya frequency function. This strengthens Schoenberg's result on Pólya frequency functions.

1. Introduction

In this paper, we introduce E-Pólya frequency functions on the real axis $\mathbb{R}$ and study their characterization properties.

Definition 1.1. A non-negative measurable function $f:\mathbb{R}\to\mathbb{R}$ with $0<\int_{\mathbb{R}} f(y)\,dy<\infty$ is said to be a Pólya frequency (PF) function if and only if for every $n=1,2,\ldots$ and any sets of increasing numbers $u_1<\cdots<u_n$, $x_1<\cdots<x_n$, the following inequality holds: $\det|f(u_i-x_j)|_{1\leqslant i,j\leqslant n}\geqslant 0$.

PF functions or more generally totally positive functions, their generalizations and applications in various areas of mathematics have attracted much attention since the late 1940s, see Karlin [3] and more recent surveys in [10] for references and discussions. In particular, approximation analysts use PF functions f to generate weak Chebyshev systems $\{f(u_i-x)\}_{i=1}^n$ of functions on $\mathbb{R}$ (see Karlin and Studden [4] and Zalik [11]).

Schoenberg [9] (see also [3, Theorem 7.3.2]) found a necessary and sufficient condition for an integrable function to be a Pólya frequency function. We present his classical result in the following equivalent form.

Theorem 1.1. *An integrable function $f:\mathbb{R}\to\mathbb{R}$ with $0<\int_{\mathbb{R}} f(y)\,dy<\infty$ is a Pólya frequency function if and only if the following conditions are satisfied*:

(i) *There exist numbers a and b, $-\infty\leqslant a<0<b\leqslant\infty$, such that the Laplace transform $\int_{\mathbb{R}} f(y)e^{-zy}\,dy$ exists in the open strip $a<\Re z<b$.*
(ii) *There exist numbers $C>0$, $\gamma\geqslant 0$, $\delta\in\mathbb{R}$, $\alpha_i\in\mathbb{R}$, $i=1,2,\ldots,$ satisfying the conditions*

$$0<\gamma+\sum_{i=1}^{\infty}\alpha_i^2<\infty,\qquad a=\max_{i,\alpha_i>0}(-1/\alpha_i),\qquad b=\min_{i,\alpha_i<0}(-1/\alpha_i),$$

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such that

$$\int_{\mathbb{R}} f(y)e^{-zy}\,dy = \frac{Ce^{\gamma z^2+\delta z}}{\prod_{i=1}^{\infty}(1+\alpha_i z)e^{-\alpha_i z}}, \qquad a < \Re z < b. \tag{1.1}$$

In this paper, we strengthen Theorem 1.1 by using weaker definitions of PF functions (so-called E-PF functions), which nevertheless allow representations like (1.1) (Theorems 2.1 and 2.2). Section 2 contains the definition of E-PF functions and statements of Theorems 2.1 and 2.2. Some properties of E-PF functions are discussed in Section 3. In particular, one more criterion for an E-PF function to be a Pólya frequency function is given (Theorem 3.1). The proofs of Theorems 2.1, 2.2, and 3.1 are given in Section 4.

In the sequel, $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ is the complex plane, $\mathbb{Z}$ the set of all integers, and $[x]$ denotes the largest integer n such that $n \leqslant x$.

2. *E*-totally positive functions

Definition 2.1. Let $E \subseteq \mathbb{R}$ be an infinite set. A non-negative measurable function $f:\mathbb{R}\to\mathbb{R}$ with $0 < \int_{\mathbb{R}} f(y)\,dy < \infty$ is said to be an E-Pólya frequency (E-PF) function if and only if for every $n = 1, 2, \ldots$ and any sets of increasing numbers $u_1 < \cdots < u_n$, $x_1 < \cdots < x_n$, such that $\{u_i\}_{i=1}^n \subset E$, $\{x_j\}_{j=1}^n \subset \mathbb{R}$, the following inequality holds: $\det|f(u_i - x_j)|_{1\leqslant i,j\leqslant n} \geqslant 0$.

Definition 2.1 is a special case of a general definition of totally positive kernels $K(u,x)$ of two variables defined on $U \times X \subseteq \mathbb{R}^2$ [3, p. 46]. Obviously, any $\mathbb{R}$-PF function is a PF function.

We call a set E *admissible* for f if f is an E-PF function and f does not allow representation (1.1). It turns out that admissible sets should satisfy fairly strict conditions. In particular, the next theorem shows that an admissible set for rapidly decreasing functions is an infinite increasing sequence (one-sided or two-sided) $\cdots < u_{-2} < u_{-1} < u_1 < u_2 < \cdots$, satisfying the condition $\lim_{s\to\infty} u_{|s|} = \infty$.

Let $\mathcal{L}$ be the class of all measurable bounded functions on $\mathbb{R}$ satisfying the condition $|f(x)| = O(\exp(-c|x|))$ as $|x| \to \infty$, where $c = c(f) > 0$ is a constant. Obviously, any $f \in \mathcal{L}$ is integrable on $\mathbb{R}$. Schoenberg [9, Lemma 2] (see also [3, Theorem 4.1.11]) showed that a PF function belongs to $\mathcal{L}$.

Theorem 2.1. *Let $E \subseteq \mathbb{R}$ be a set with an accumulation point. Then every E-PF function $f \in \mathcal{L}$ is a PF function.*

Remark 2.1. Note that we use the term "a Pólya frequency function" after Schoenberg [8,9], while Karlin's definition of Pólya frequency functions [3, p. 332] defers from ours in that he does not require that f be integrable.

If E contains a symmetric (about the origin) admissible sequence, then there is a lower restriction on increase of its elements with positive indices.

Theorem 2.2. *Let E contain an increasing sequence $\{u_s\}_{|s|=1}^{\infty}$ satisfying the following conditions*:

(i) $u_{-s} = -u_s < 0$, $s = 1, 2, \ldots$.
(ii) $\lim_{s\to\infty} u_s = \infty$.
(iii) $u_s = o(\sqrt{s})$ *as* $s \to \infty$.

Then every E-PF function $f \in \mathcal{L}$ is a PF function.

Remark 2.2. It follows from Theorems 1.1, 2.1, and 2.2 that if E satisfies conditions of Theorem 2.1 or Theorem 2.2, then $f \in \mathcal{L}$ with $0 < \int_{\mathbb{R}} f(y)\,dy < \infty$ is an E-PF function if and only if f satisfies conditions (i) and (ii) of Theorem 1.1.

Remark 2.3. It follows immediately from the basic composition formula [3, p. 17] that for a PF function h and any E-PF function $f \in \mathcal{L}$, the convolution $g(x) = h * f := \int_{\mathbb{R}} h(t) f(x-t)\,dt$ is an E-PF function. Similarly, if $\{h(k)\}_{k=-\infty}^{\infty}$ is a two-sided PF sequence (see [3, p. 419]) with $0 < \sum_{k=-\infty}^{\infty} h(k) < \infty$ and if $f \in \mathcal{L}$ is a $\mathbb{Z}$-PF function, then the discrete convolution $g(x) = h \circledast f := \sum_{k=-\infty}^{\infty} h(k) f(x-k)$ is a $\mathbb{Z}$-PF function. Indeed, it suffices to apply the basic composition formula [3, p. 17] with the measure σ supported in $\mathbb{Z}$, $\sigma(k) = 1$, $k \in \mathbb{Z}$, to $g(u-x) = \sum_{m=-\infty}^{\infty} h(u-m) f(m-x)$.

The following example shows that $\mathbb{Z}$ is admissible for some functions from $\mathcal{L}$.

Example 2.1. Let $E = \mathbb{Z}$ and let

$$f_\psi(y) := \begin{cases} \psi(y), & y \in (0,1], \\ 0, & \text{otherwise}, \end{cases}$$

where $\psi : (0,1] \to [0,\infty)$ is a bounded measurable function on $(0,1]$ with $0 < \int_0^1 \psi(x)\,dx < \infty$. Then $f_\psi \in \mathcal{L}$ and by Theorem 1.1, f_ψ is not a PF function since $\int_{\mathbb{R}} f_\psi(y)e^{-zy}\,dy$ is an entire function of exponential type but not $e^{\delta z}$, $\delta \in \mathbb{R}$. However, f_ψ is a $\mathbb{Z}$-PF function. Indeed, for $n = 1,2,\ldots$, any integers $k_1 < \cdots < k_n$, and real numbers $x_1 < \cdots < x_n$, we have $\det|f(k_i - x_j)|_{1\leqslant i,j\leqslant n} = 0$, excluding possibly the case of $k_i = [x_i]+1$, $1 \leqslant i \leqslant n$, when $\det|f(k_i - x_j)|_{1\leqslant i,j\leqslant n} = \prod_{i=1}^n \psi([x_i]+1-x_i) \geqslant 0$. Therefore, f_ψ is a $\mathbb{Z}$-PF function. In addition, note that functions $h_1 * f_\psi$ and $h_2 \circledast f_\psi$ belong to $\mathcal{L}$ (see [3, Chapters 7 and 8]) and according to Remark 2.3, they are $\mathbb{Z}$-PF functions. Here, h_1 is a PF function and $\{h_2(k)\}_{k=-\infty}^{\infty}$ a two-sided PF sequence.

3. Properties of E-PF functions

To prove the theorems, we shall need some properties of E-PF functions.

Proposition 3.1. *Let $E \subseteq \mathbb{R}$ be an infinite set. Then the following statements hold*:

(a) *If f is an E-PF function, then for any set $E_1 \subseteq E$, f is an E_1-PF function.*
(b) *If f is an E-PF function, then for any $\alpha \in \mathbb{R}$, f is an $(E+\alpha)$-PF function.*
(c) *Let $f \in \mathcal{L}$ be an E-PF function and let h be a locally integrable function on $\mathbb{R}$. If the convolution $g(u) = \int_{\mathbb{R}} h(x)f(u-x)\,dx$ exists for every $u \in E$, then*

$$S\big[g(u), E\big] \leqslant S\big[h(x), \mathbb{R}\big],$$

where $S[\varphi(t), A]$ denotes a number of sign changes of $\varphi(t)$ on a set $A \in \mathbb{R}$ (see [3, Section 3] *for definitions).*

Statements (a) and (b) of the proposition follow immediately from Definition 2.1, while statement (c) is a special case of a more general result [3, Theorem 5.3.1(i)].

Next, we discuss some properties of polynomials and the Laplace transform of an E-PF function $f \in \mathcal{L}$. We first note that all the moments

$$\mu_k := \int_{\mathbb{R}} f(y)y^k\,dy, \quad k = 0,1,\ldots,$$

exist since $f \in \mathcal{L}$. In addition, $f \geqslant 0$ on $\mathbb{R}$, and we may assume without loss of generality that $0 < \mu_0 = \int_{\mathbb{R}} f(y)\,dy < \infty$. Then the convolution

$$g_n(u) := \int_{\mathbb{R}} h_n(x)f(u-x)\,dx \tag{3.1}$$

exists for every polynomial h_n of degree n and every $u \in \mathbb{R}$.

Further, the Laplace integral

$$F(z) := \int_{\mathbb{R}} e^{-zy}f(y)\,dy = \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k}{k!} z^k$$

converges in some strip $\{z \in \mathbb{C}\colon -\infty \leqslant a \leqslant \Re z \leqslant b \leqslant \infty\}$ containing the origin. Since $F(0) = \mu_0 > 0$, the reciprocal

$$\Phi(z) := \frac{1}{F(z)} = \sum_{m=0}^{\infty} \frac{\lambda_m}{m!} z^m \tag{3.2}$$

exists in some non-degenerate neighborhood of the origin in $\mathbb{C}$.

Then the following proposition is valid.

Proposition 3.2. *Let $E \subseteq \mathbb{R}$ be an infinite set. Next, let $f \in \mathcal{L}$ be an E-PF function and let g_n be defined by the transformation* (3.1)*. Then for $n = 1,2,\ldots$, the following statements hold*:

(a) *If h_n is a real polynomial of exact degree n, then g_n is a real polynomial of exact degree n.*
(b) *For any real polynomial g_n of exact degree n, there exists the only real polynomial h_n of exact degree n such that* (3.1) *holds.*
(c) *The polynomial h_n from statement* (b) *can be found by the formula*

$$h_n(x) = \sum_{k=0}^{\infty} \frac{\lambda_k}{k!}(g_n)^{(k)}(x) = \sum_{k=0}^{n} \frac{\lambda_k}{k!}(g_n)^{(k)}(x), \tag{3.3}$$

where $\{\lambda_m\}_{m=0}^{\infty}$ is the sequence of coefficients from (3.2)*.*

(d) $S[g_n(u), E] \leqslant S[h_n(x), \mathbb{R}]$.
(e) *Let g_n be a real polynomial of exact degree n having n zeros $u_k \in E$, $1 \leqslant k \leqslant n$, where $u_1 > \cdots > u_n$, and let h_n be the polynomial from statement* (c)*. In addition, we assume that either* $\inf E < u_n$ *or* $\sup E > u_1$*. Then h_n has n real zeros.*

Proof. Statements (a) and (b) of the proposition follow from fact that there is the linear non-singular transformation among the coefficients of h_n and g_n. Indeed, let

$$h_n(x) = \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k}, \qquad g_n(u) = \sum_{k=0}^{n} b_k \binom{n}{k} u^{n-k}.$$

Then (see [9, p. 346] or [3, p. 343])

$$b_k = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} \mu_{k-s} a_s, \quad 0 \leqslant k \leqslant n. \tag{3.4}$$

Since the determinant of the transformation matrix is equal to $\mu_0^n > 0$, statements (a) and (b) follow. Statement (c) is proved in [3, p. 353] (see also [3, p. 344]). Statement (d) follows from Proposition 3.1(c).

It remains to prove statement (e). Let $g_n(u) = b_0(u-u_1)\dots(u-u_n)$, where $u_1 > \cdots > u_n$ and $u_k \in E$, $1 \leqslant k \leqslant n$. Without loss of generality we can assume that $\inf E < u_n$. Then there exists $u_{n+1} \in E$ such that $u_{n+1} < u_n$. Let us consider a polynomial $g_{n,s}(u) := b_0(u-u_{1,s})\dots(u-u_{n,s})$, where

$$u_1 > u_{1,s} > u_2 > \cdots > u_{n-1} > u_{n-1,s} > u_n > u_{n,s} > u_{n+1}, \quad s = 1, 2, \dots,$$

and $\lim_{s\to\infty} u_{k,s} = u_k$, $1 \leqslant k \leqslant n$. Since $\operatorname{sgn}(g_{n,s}(u_k)) = \operatorname{sgn}(-g_{n,s}(u_{k+1})) \neq 0$, $1 \leqslant k \leqslant n$, we have $S[g_{n,s}, E] = n$, $s = 1, 2, \dots$. Next by statement (b) of this proposition, there exists a sequence of real polynomials $h_{n,s}$ of exact degree n with the leading coefficient b_0/μ_0 such that

$$g_{n,s}(u) := \int_{\mathbb{R}} h_{n,s}(x) f(u-x)\,dx, \quad s = 1, 2, \dots. \tag{3.5}$$

Moreover by statement (d), all $h_{n,s}$, $s = 1, 2, \dots$, have only real zeros. Furthermore, it is easy to see that $\lim_{s\to\infty} g_{n,s}(u) = g_n(u)$ uniformly on any compact in $\mathbb{R}$. Hence by linear system (3.4), $\lim_{s\to\infty} h_{n,s}(x) = h_n(x)$ uniformly on any compact in $\mathbb{R}$, where h_n is polynomial (3.3). Since the limit polynomial h_n has only real zeros, statement (e) follows. □

The following example shows that Proposition 3.2(e) cannot be extended to the case of multiple zeros $u_1 \geqslant \cdots \geqslant u_n$ belonging to $E = \mathbb{Z}$. Note that in the case of $E = \mathbb{R}$, there is an analogue of Proposition 3.2(e) for multiple zeros (see [9, Lemma 7] or [3, Theorem 5.4.6]). A more general approach to zero-increasing transformations was discussed by Pinkus in survey [6], and the latest result about transformation (3.3) was obtained by Carnicer, Peña, and Pinkus [1, Theorem 1].

Example 3.1. Let

$$f_1(y) := \begin{cases} 1, & y \in (0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

Since f_1 is a special case of a function from Example 2.1, f_1 is not a PF function but f_1 is a $\mathbb{Z}$-PF function. Let n be a positive integer and let h_n be the polynomial of exact degree n such that

$$u^n = \int_{\mathbb{R}} f(y) h_n(u-y)\,dy. \tag{3.6}$$

Then we state that for $n \geqslant 3$ a number of real zeros of h_n counted with their multiplicities is less than n. Indeed, solving linear system (3.4) for $n = 3$, we have $h_3(t) = (1/2)(2t^3 + 3t^2 + t + 1)$. It is easy to see that h_3 has only one real zero $t = -1.39816\dots$. Next if $n > 3$, then differentiating both sides of (3.6) $n-3$ times and taking account of Proposition 3.2(b), we arrive at the relation $h_3(t) = (6/n!)h_n^{(n-3)}(t)$. Hence if we assume that h_n has n real zeros counted with their multiplicities, then by Rolle's theorem, h_3 has three real zeros. This contradiction proves the statement.

The following result shows that Example 3.1 is typical in a sense for symmetric unbounded sets E admissible for f.

Theorem 3.1. *A function f is a PF function if and only if the following conditions are satisfied*:

(i) $f \in \mathcal{L}$.
(ii) *There exists a symmetric increasing sequence $E = \{u_s\}_{|s|=1}^{\infty}$ with $\lim_{s\to\infty} u_s = \infty$ such that*

(1) *f is an E-PF function*;
(2) *for any real polynomial g_n of exact degree n having n real zeros on E counted with their multiplicities, the polynomial h_n defined by* (3.3) *has n real zeros counted with their multiplicities, $n = 1, 2, \ldots$.*

Remark 3.1. It follows from Theorem 3.1 that if $f \in \mathcal{L}$ is not a PF function but f is an E-PF function, where E satisfies condition (ii) of Theorem 3.1, then there exist an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ and a sequence of polynomials $\{g_{n_k}\}_{k=1}^{\infty}$ of exact degree n_k such that g_{n_k} has n_k real zeros on E counted with their multiplicities while a number of real zeros of h_{n_k} is less than n_k, $k = 1, 2, \ldots$. Example 3.1 provides construction of these polynomials for $E = \mathbb{Z}$ and $n_k = k$, $k = 1, 2, \ldots$.

4. Proofs of Theorems 2.1, 2.2, and 3.1

The proofs of these theorems are based on Schoenberg's idea [9] of using representations of entire functions which are limits, uniform in every finite domain, of real polynomials with only real zeros [2,5,7].

Proof of Theorem 2.1. Let $f \in \mathcal{L}$ be an E-PF function, where E has an accumulation point μ. Taking account of statements (a) and (b) of Proposition 3.1, we can assume without loss of generality that $\mu = 0$ and $E = \{u_s\}_{s=1}^{\infty}$ is a monotone (say, decreasing) sequence of real numbers such that $\lim_{s\to\infty} u_s = 0$.

Next, we consider the polynomial

$$g_n(u) := \prod_{s=1}^{n}(u - u_s) = \sum_{l=0}^{n} c_l u^l, \qquad c_n = 1.$$

Using statements (c) and (e) of Proposition 3.2, we see that there exists a real polynomial h_n of exact degree n having all real zeros, and its analytic expression is given by the formula

$$h_n(x) = \sum_{k=0}^{n} \frac{\lambda_k}{k!}(g_n)^{(k)}(x) = \sum_{k=0}^{n} \frac{\lambda_k}{k!}\left(\sum_{l=0}^{n} c_l x^l\right)^{(k)}. \tag{4.1}$$

We recall that $\{\lambda_m\}_{m=1}^{\infty}$ is a sequence of coefficients in the expansion

$$\left(\int_{\mathbb{R}} f(y)e^{-zy}\,dy\right)^{-1} = \sum_{m=0}^{\infty} \frac{\lambda_m}{m!} z^m, \tag{4.2}$$

which converges in some neighborhood of the origin in $\mathbb{C}$.

Further, it follows from (4.1) that

$$h_n(x) = \sum_{m=0}^{n}\left(\sum_{k=0}^{n-m} \lambda_k c_{m+k}\binom{m+k}{k}\right)x^m$$

and setting $Q_n(x) := x^n h_n(1/x)$, we have

$$Q_n(x/n) = \sum_{m=0}^{n} n^{-m}\left(\sum_{k=0}^{m} \lambda_k c_{n-m+k}\binom{n-m+k}{k}\right)x^m. \tag{4.3}$$

Next, we shall show that

$$\lim_{n\to\infty} Q_n(z/n) = \sum_{m=0}^{\infty} \frac{\lambda_m}{m!} z^m \tag{4.4}$$

uniformly in some neighborhood V of the origin in $\mathbb{C}$. Indeed, it follows from (4.3) that

$$\left|\sum_{m=0}^{\infty} \frac{\lambda_m}{m!} z^m - Q_n(z/n)\right| \leqslant \sum_{m=0}^{n} \frac{|\lambda_m|}{m!}\left(1 - \frac{n!}{(n-m)!n^m}\right)|z|^m + \sum_{m=n+1}^{\infty} \frac{|\lambda_m|}{m!}|z|^m + \sum_{m=1}^{n} n^{-m}\left|\sum_{k=0}^{m-1} \lambda_k c_{n-m+k}\binom{n-m+k}{k}\right||z|^m. \tag{4.5}$$

Further, for prescribed $\varepsilon > 0$, we choose $N = N(\varepsilon)$ such that

$$\sum_{m=N+1}^{\infty} \frac{|\lambda_m|}{m!}|z|^m < \varepsilon, \quad z \in V. \tag{4.6}$$

Hence for $n > n_1(\varepsilon) > N$,

$$\sum_{m=0}^{n} \frac{|\lambda_m|}{m!}\left(1 - \frac{n!}{(n-m)!n^m}\right)|z|^m + \sum_{m=n+1}^{\infty} \frac{|\lambda_m|}{m!}|z|^m \leqslant \sum_{m=0}^{N} \frac{|\lambda_m|}{m!}\left(1 - \frac{n!}{(n-m)!n^m}\right)|z|^m + 3\sum_{m=N+1}^{\infty} \frac{|\lambda_m|}{m!}|z|^m < 4\varepsilon, \tag{4.7}$$

since for a fixed N,

$$\lim_{n\to\infty}\left(1 - \frac{n!}{(n-m)!n^m}\right) = 0, \quad 0 \leqslant m \leqslant N.$$

It remains to estimate

$$I_n(z) := \sum_{m=1}^{n} n^{-m}\left|\sum_{k=0}^{m-1} \lambda_k c_{n-m+k}\binom{n-m+k}{k}\right||z|^m. \tag{4.8}$$

Using elementary relations

$$|c_{n-m+k}| \leqslant \left(\sum_{s=1}^{n} u_s\right)^{m-k}, \quad 0 \leqslant k \leqslant m-1, \qquad \lim_{n\to\infty} n^{-1}\sum_{s=1}^{n} u_s = 0,$$

we obtain for $n > n_2(\varepsilon)$,

$$I_n(z) \leqslant \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \frac{|\lambda_k|}{k!}\left(\frac{\sum_{s=1}^{n} u_s}{n}\right)^{m-k}|z|^m < \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \frac{|\lambda_k|}{k!}\varepsilon^{m-k}|z|^m = \varepsilon|z|\sum_{k=0}^{\infty} \frac{|\lambda_k|}{k!}\sum_{d=k}^{\infty}\varepsilon^{d-k}|z|^d = \frac{\varepsilon|z|}{1-\varepsilon|z|}\sum_{k=0}^{\infty}\frac{|\lambda_k|}{k!}|z|^k.$$

Hence we get for small enough ε,

$$I_n(z) < C\varepsilon, \quad z \in V, \ n > n_2(\varepsilon), \tag{4.9}$$

where C is independent of z, n, and ε. Combining (4.5), (4.7), (4.8), and (4.9), we arrive at (4.4).

Thus (4.2) and (4.4) imply the limit relation

$$\Phi(z) := \left(\int_{\mathbb{R}} f(y)e^{-zy}\,dy\right)^{-1} = \lim_{n\to\infty} Q_n(z/n), \tag{4.10}$$

which holds uniformly in V.

To complete the proof, we recall that h_n is a real polynomial of exact degree n having only real zeros. Then $Q_n(x) = x^n h_n(1/x)$ is a real polynomial having only real zeros (if Q_n is not a constant) of degree $n-d$, where d is a multiplicity of 0 as a zero of h_n. Therefore the polynomial $Q_n(x/n)$ has only real zeros. Then by (4.10), the function Φ is a limit, uniform in some neighborhood of the origin in $\mathbb{C}$, of real polynomials with only real zeros. By a classic theorem of Laguerre [5] and Pólya [7] (see also [2, Chapter IV]), the analytic extension of Φ is an entire function, and integral representation (1.1) holds with the constants satisfying condition (ii) of Theorem 1.1. Finally, it follows from the sufficient part of Theorem 1.1 that f is a PF function. □

Remark 4.1. Below we outline a slightly different proof of Theorem 2.1. Assuming that $E = \{u_s\}_{s=1}^{\infty}$, where $u_s \downarrow 0$ as $s \to \infty$, we consider a sequence of polynomials $g_{n,m}(u) := \prod_{s=1}^{n}(u - u_{s+m})$, which converges to u^n as $m \to \infty$ uniformly in any compact of $\mathbb{R}$. The corresponding sequence of $h_{n,m}$, satisfying (3.5), converges to a polynomial h_n, which has only real zeros. Finally, Schoenberg's argument from [9, p. 353] completes the proof. This proof appears to be shorter, however, the proof of Theorem 2.1 can be extended to some other situations (see the proofs of Theorems 2.2 and 3.1).

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.1. Let $f \in \mathcal{L}$ be an E-PF function, where $E = \{u_s\}_{|s|=1}^{\infty}$ is a symmetric increasing sequence with $u_s \to \infty$ and $u_s = o(\sqrt{s})$ as $s \to \infty$. Let us consider the polynomial

$$g_{2n}(u) := \prod_{s=1}^{n}\left(u^2 - u_s^2\right) = \sum_{l=0}^{2n} c_l u^l,$$

where $c_l = 0$ if l is odd, $0 \leqslant l \leqslant 2n$, and $c_{2n} = 1$. Then by Proposition 3.2, relations (4.1), (4.2), and (4.3) hold with n replaced by $2n$, that is,

$$Q_{2n}\big(x/(2n)\big) = \sum_{m=0}^{2n} (2n)^{-m} \left(\sum_{k=0}^{m} \lambda_k c_{2n-m+k} \binom{2n-m+k}{k} \right) x^m. \tag{4.11}$$

Next, we shall show that

$$\lim_{n\to\infty} Q_{2n}\left(\frac{z}{2n}\right) = \sum_{m=0}^{\infty} \frac{\lambda_m}{m!} z^m \tag{4.12}$$

uniformly in some neighborhood V of the origin in $\mathbb{C}$. Indeed, it follows from (4.11) that

$$\begin{aligned}\left| \sum_{m=0}^{\infty} \frac{\lambda_m}{m!} z^m - Q_{2n}\left(\frac{z}{2n}\right) \right| &\leqslant \sum_{m=0}^{2n} \frac{|\lambda_m|}{m!} \left(1 - \frac{(2n)!}{(2n-m)!(2n)^m}\right) |z|^m + \sum_{m=2n+1}^{\infty} \frac{|\lambda_m|}{m!} |z|^m \\ &\quad + \sum_{m=1}^{2n} (2n)^{-m} \left| \sum_{k=0}^{m-1} \lambda_k c_{2n-m+k} \binom{2n-m+k}{k} \right| |z|^m. \end{aligned} \tag{4.13}$$

Further, for prescribed $\varepsilon > 0$, we choose $N = N(\varepsilon)$ such that (4.6) holds. Hence for $n > n_1(\varepsilon) > N/2$, the first two sums in the right-hand side of (4.13) are bounded by 4ε (like in (4.7)). It remains to estimate

$$\begin{aligned} I_{2n}(z) &:= \sum_{m=1}^{2n} (2n)^{-m} \left| \sum_{k=0}^{m-1} \lambda_k c_{2n-m+k} \binom{2n-m+k}{k} \right| |z|^m \\ &\leqslant \sum_{d=1}^{n} (2n)^{-2d} \left| \sum_{l=0}^{d-1} \lambda_{2l} c_{2n-2d+2l} \binom{2n-2d+2l}{2l} \right| |z|^{2d} \\ &\quad + \sum_{d=2}^{n} (2n)^{-2d+1} \left| \sum_{l=1}^{d-1} \lambda_{2l-1} c_{2n-2d+2l} \binom{2n-2d+2l}{2l-1} \right| |z|^{2d-1}. \end{aligned} \tag{4.14}$$

Using relations

$$|c_{2n-2d+2l}| \leqslant \left(\sum_{s=1}^{n} u_s^2 \right)^{d-l}, \quad 0 \leqslant l \leqslant d-1, \tag{4.15}$$

$$\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} u_s^2 = 0, \tag{4.16}$$

we obtain from (4.14) for $n > n_2(\varepsilon)$ and $z \in V$,

$$I_n(z) < \sum_{d=1}^{\infty} \sum_{l=0}^{d-1} \frac{|\lambda_{2l}|}{(2l)!} \varepsilon^{2d-2l} |z|^{2d} + \sum_{d=2}^{\infty} \sum_{l=1}^{d-1} \frac{|\lambda_{2l-1}|}{(2l-1)!} \varepsilon^{2d-2l} |z|^{2d-1} \leqslant \frac{\varepsilon^2 |z|^2}{1 - \varepsilon^2 |z|^2} \sum_{m=0}^{\infty} \frac{|\lambda_m|}{m!} |z|^m \leqslant C\varepsilon^2,$$

where C is independent of z, n, and ε. Therefore (4.12) follows.

Next by Proposition 3.2, $Q_{2n}(x/(2n))$ has only real zeros, and the limit relation

$$\Phi(z) := \left(\int_{\mathbb{R}} f(y) e^{-zy}\, dy \right)^{-1} = \lim_{n\to\infty} Q_n\big(z/(2n)\big)$$

holds uniformly in some neighborhood of the origin in $\mathbb{C}$. Therefore by the theorem of Laguerre [5] and Pólya [7] (see also [2, Chapter IV]), the analytic extension of Φ is an entire function and (1.1) holds with the constants satisfying condition (ii) of Theorem 1.1. Hence f is a PF function. □

Proof of Theorem 3.1. The necessity of the theorem follows immediately from properties of PF functions. Property (i) of PF functions was proved in [9, Lemma 2], while conditions (ii)(1) and (ii)(2) are satisfied for any E satisfying condition (ii). In particular, an inequality between numbers of zeros of g_n and h_n (see [9, Lemma 7] or [3, Theorem 5.4.6]) shows that condition (ii)(2) is satisfied.

Let now conditions (i), (ii)(1), and (ii)(2) be satisfied. For any $N = 1, 2, \ldots$ we consider the polynomial

$$g_{2n}(u) := \prod_{s=1}^{N} \left(u^2 - u_s^2\right)^{p_s} = \sum_{l=0}^{2n} c_l u^l,$$

where $p_s := ([|u_s|] + 1)^3$, $1 \leqslant |s| \leqslant N$, and $n := \sum_{s=1}^{N} p_s$.

Then the proof of the fact that f is a PF function is similar to that of Theorem 2.2, if we replace relations (4.15) and (4.16) with

$$|c_{2n-2d+2l}| \leqslant \left(\sum_{s=1}^{N} p_s u_s^2\right)^{d-l}, \quad 0 \leqslant l \leqslant d-1,$$

and

$$\lim_{n\to\infty} n^{-2} \sum_{s=1}^{N} p_s u_s^2 \leqslant \lim_{N\to\infty} \frac{\sum_{s=1}^{N} (u_s + 1)^5}{\sum_{s=1}^{N} u_s^6} = 0,$$

respectively. □

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